

The Limiting Distribution of the Number of Block Pairs in Type B Set Partitions

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Abstract

It is a classical result of Harper that the limiting distribution of the number of blocks in partitions of the set $\{1, 2, \dots, n\}$ is normal. In this paper, using the saddle point method we prove the normality of the limiting distribution of the number of block pairs in set partitions of type B_n . Moreover, we obtain that the limiting distribution of the number of block pairs in B_n -partitions without zero-block is also normal.

1 Introduction

This paper is concerned with the limiting distribution of the number of block pairs of type B_n set partitions. For ordinary set partitions, Harper [11] has established the normality of the limiting distribution of the number of blocks in partitions of the set $\{1, 2, \dots, n\}$. For the asymptotic behavior concerning with the ordinary set partitions, see [6, 9, 10, 14, 17, 18]. For the study on the limiting distribution of other combinatorial objects, see Flajolet and Sedgewick's book [7] for instance.

The lattice of ordinary set partitions can be regarded as the intersection lattice for the hyperplane arrangement corresponding to the root system of type A , see Björner and Brenti [3] or Humphreys [13]. From this point of view, type B set partitions are a generalization of ordinary partitions, see Reiner [19]. To be more precise, ordinary set partitions encode the intersections of hyperplanes in the hyperplane arrangement for the type A root system, while the intersections of subsets of hyperplanes from the type B hyperplane arrangement can be encoded by type B set partitions, see Björner and Wachs [4]. A type B_n set partition is a partition π of the set

$$\{1, 2, \dots, n, -1, -2, \dots, -n\}$$

such that for any block B of π , $-B$ is also a block of π , and there is at most one block, called zero-block, satisfying $B = -B$. We call $(B, -B)$ a block pair of π if B is not a zero-block.

Let $M_{n,k}$ be the number of B_n -partitions with k block pairs. It is easy to deduce the recurrence relation

$$M_{n,k} = M_{n-1,k-1} + (2k+1)M_{n-1,k}. \quad (1.1)$$

The main result of this paper is to derive the limiting distribution of the number of block pairs in B_n -partitions based on the above recurrence formula. Let ξ_n be the random variable of the number of block pairs in B_n -partitions. We shall prove that the limiting distribution of ξ_n is normal by using the saddle point method, which was introduced by Schrödinger [21], see also [2, 5, 8, 16].

This paper is organized as follows. In Section 2, we present some facts about the saddle point of the generating function of the number of B_n -partitions. Section 3 is devoted to deduce the normality of the limiting distribution of ξ_n . Using the same technique, we obtain the normality of the limiting distribution of the number of block-pairs in B_n -partitions without zero-block.

2 Preliminary lemmas

Let M_n be the number of B_n -partitions. In this section, we give some lemmas which will be used to derive an approximate formula for M_n .

Let $N_{n,k}$ be the number of B_n -partitions with k block pairs but with no zero-block. Denote by N_n the number of B_n -partitions without zero-block. It is easy to see that

$$N_{n,k} = 2^{n-k} S(n, k).$$

Since

$$\sum_n S(n, k) \frac{x^n}{n!} = \frac{1}{k!} (e^x - 1)^k, \quad (2.1)$$

see Stanley [22, page 34], we find that

$$F_N(z) = \sum_{n \geq 0} N_n \frac{z^n}{n!} = \exp\left(\frac{e^{2z} - 1}{2}\right). \quad (2.2)$$

It is also easy to see that

$$M_n = \sum_k \binom{n}{k} N_k.$$

It follows from (2.2) that

$$F_M(z) = \sum_{n \geq 0} M_n \frac{z^n}{n!} = \exp\left(\frac{e^{2z} - 1}{2} + z\right). \quad (2.3)$$

The saddle point of $F_M(z)$ is defined to be the value z that minimizes $z^{-n} F_M(z)$, i.e., the unique positive solution r_1 of the equation

$$r_1(e^{2r_1} + 1) = n. \quad (2.4)$$

Similarly, the saddle point of $F_N(z)$ is the unique positive solution r_0 of the equation

$$r_0 e^{2r_0} = n. \quad (2.5)$$

For convenience, we consider the equation

$$r (e^{2r} + c) = n. \quad (2.6)$$

It reduces to (2.4) when $c = 1$, and to (2.5) when $c = 0$. It is easy to deduce the following approximation for the unique positive solution r of (2.6).

Lemma 2.1. *Let c be a nonnegative integer. Let $r > 0$ be the unique positive solution of equation (2.6). Then we have*

$$\begin{aligned} r &= \frac{\log n}{2} \left(1 + O\left(\frac{\log \log n}{\log n} \right) \right), \\ e^{2r} &= \frac{2n}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n} \right) \right). \end{aligned}$$

We will also need the following lemma.

Lemma 2.2. *Let $h(x)$ be a continuous function defined on the closed interval $[a, b]$. Suppose that $h''(x)$ exists in the open interval (a, b) . Then for any $c \in (a, b)$, there exists $s \in (a, b)$ such that*

$$\frac{h(a)}{(a-b)(a-c)} + \frac{h(b)}{(b-a)(b-c)} + \frac{h(c)}{(c-a)(c-b)} = \frac{h''(s)}{2}. \quad (2.7)$$

Proof. Let

$$\begin{aligned} f_1(x) &= (a-b)h(x) + (b-x)h(a) + (x-a)h(b), \\ g_1(x) &= (a-b)(b-x)(x-a). \end{aligned}$$

Then the left hand side of (2.7) becomes $f_1(c)/g_1(c)$. Note that $f_1(a) = g_1(a) = 0$. By Cauchy's mean value theorem, there exists $s_1 \in (a, c)$ such that

$$\frac{f_1(c)}{g_1(c)} = \frac{f_1'(s_1)}{g_1'(s_1)} = \frac{f_2(a) - f_2(b)}{g_2(a) - g_2(b)},$$

where $f_2(x) = h'(s_1)x - h(x)$ and $g_2(x) = x^2 - 2s_1x$. Again, by Cauchy's mean value theorem, there exist $s_2 \in (a, b)$ and $s \in (a, b)$ such that

$$\frac{f_1(c)}{g_1(c)} = \frac{f_2'(s_2)}{g_2'(s_2)} = \frac{h'(s_1) - h'(s_2)}{2s_1 - 2s_2} = \frac{h''(s)}{2}.$$

This completes the proof. ■

Lemma 2.3. *Let c be a nonnegative integer, and $f(x) = x(e^{2x} + c)$. Suppose that t_i ($i = 0, 1, 2$) is the unique positive number such that $f(t_i) = n + i$. Then we have*

$$t_1 - t_0 = \frac{1}{2n} - \frac{1}{4nt_0} + O\left(\frac{1}{n \log^2 n}\right), \quad (2.8)$$

$$t_2 - t_1 = \frac{1}{2n} - \frac{1}{4nt_0} + O\left(\frac{1}{n \log^2 n}\right), \quad (2.9)$$

$$2t_1 - t_0 - t_2 = \frac{1}{n^2} + O\left(\frac{1}{n^2 \log n}\right),$$

$$\frac{1}{t_0} + \frac{1}{t_2} - \frac{2}{t_1} = O\left(\frac{1}{n^2 \log^2 n}\right).$$

Proof. We first consider (2.8). By Cauchy's mean value theorem, there exists t such that $t_0 < t < t_1$ and

$$f(t_1) - f(t_0) = (t_1 - t_0)f'(t).$$

Since $f(t_1) - f(t_0) = 1$ and $f'(t) = (2t + 1)e^{2t} + c$, we have

$$\frac{1}{(2t_1 + 1)e^{2t_1} + c} \leq t_1 - t_0 \leq \frac{1}{(2t_0 + 1)e^{2t_0} + c}. \quad (2.10)$$

It can be seen that both $\frac{1}{(2t_1 + 1)e^{2t_1} + c}$ and $\frac{1}{(2t_0 + 1)e^{2t_0} + c}$ have the same estimate

$$\frac{1}{2n} - \frac{1}{4nt_0} + O\left(\frac{1}{n \log^2 n}\right).$$

It follows that $t_1 - t_0$ also has the above estimate. Similarly, one can prove (2.9). The last two approximations are consequences of Lemma 2.2. ■

3 The limiting distribution

Recall that ξ_n is the random variable of the number of block pairs in a B_n -partition. Denote by $E(\xi_n)$ the expectation of ξ_n , and $V(\xi_n)$ the variance of ξ_n . Below is the main result of this paper.

Theorem 3.1. *The limiting distribution of the random variable ξ_n is normal. In other words, the random variable*

$$\frac{\xi_n - E(\xi_n)}{\sqrt{V(\xi_n)}}$$

has an asymptotically standard normal distribution as n tends to infinity.

There are various sufficient conditions on a random variable which ensures a normal limiting distribution, see Sachkov [20]. Let η_n be a random variable of certain statistic of some combinatorial objects on a set A_n . Let $a_n(k)$ be the number of elements of A_n with the statistic equal to k . Consider the polynomial

$$P_n(x) = \sum_k a_n(k) x^k.$$

The following criterion was used by Harper [11], see also Bender [1].

Proposition 3.2. *The limiting distribution of η_n is normal, if the $P_n(x)$ distinct real roots and the variance of $a_n(k)$ tends to infinity as $n \rightarrow \infty$.*

We shall prove Theorem 3.1 with the aid of Proposition 3.2. Recall that $M_{n,k}$ is the number of B_n -partitions with k block pairs. Consider the polynomial

$$M_n(x) = \sum_k M_{n,k} x^k. \quad (3.1)$$

For example,

$$\begin{aligned} M_1(x) &= 1 + x, \\ M_2(x) &= 1 + 4x + x^2, \\ M_3(x) &= 1 + 13x + 9x^2 + x^3. \end{aligned}$$

Theorem 3.3. *For any $n \geq 1$, the polynomial $M_n(x)$ has n distinct real roots.*

Proof. The proof is similar to the proof of Harper for ordinary partitions. We prove it by induction on n . It is clear that the theorem holds for $n = 1, 2$. We assume that it holds for all $n \leq m - 1$, where $m \geq 3$. Let

$$G_n(x) = \sqrt{x} e^{\frac{x}{2}} M_n(x). \quad (3.2)$$

Differentiating $M_n(x)$ with respect to x and using the recurrence (1.1), we obtain that

$$M_n(x) = (1 + x)M_{n-1}(x) + 2xM'_{n-1}(x). \quad (3.3)$$

Multiplying both sides of (3.3) by $\sqrt{x} e^{\frac{x}{2}}$ yields

$$G_n(x) = 2xG'_{n-1}(x). \quad (3.4)$$

By the induction hypothesis, we may assume that $M_{m-1}(x)$ has roots x_1, x_2, \dots, x_{m-1} where $x_1 < x_2 < \dots < x_{m-1} < 0$. Observe that

$$\lim_{x \rightarrow -\infty} G_n(x) = 0.$$

From (3.2) it can be seen that $G_{m-1}(x)$ has $m+1$ roots

$$-\infty, x_1, x_2, \dots, x_{m-1}, 0.$$

By Rolle's theorem, in each of the m open intervals

$$(-\infty, x_1), (x_1, x_2), \dots, (x_{m-1}, 0),$$

there exists a point y such that $G'_{m-1}(y) = 0$. Suppose that

$$G'_{m-1}(y_1) = G'_{m-1}(y_2) = \dots = G'_{m-1}(y_m) = 0,$$

where $y_1 < y_2 < \dots < y_m < 0$. By (3.4), the function $G_m(x)$ has $m+2$ roots

$$-\infty, y_1, y_2, \dots, y_m, 0.$$

Because of (3.2), we see that y_1, y_2, \dots, y_m are m distinct negative roots of $M_m(x)$. This completes the proof. \blacksquare

It should be mentioned that Theorem 3.3 can also be deduced from the criteria of Liu and Wang [15]. The following theorem gives an estimate of M_n .

Theorem 3.4. *We have*

$$M_n = \frac{1}{\sqrt{2r_1+1}} \exp\left(2nr_1 - n + \frac{n}{2r_1} + 2r_1 - 1\right) \left[1 + O\left(\frac{\log^{7/2} n}{\sqrt{n}}\right)\right], \quad (3.5)$$

where r_1 is the unique positive solution of the equation $r_1(e^{2r_1} + 1) = n$.

Proof. Let $r = r_1$. Applying Cauchy's formula and the generating function (2.3), we have

$$\frac{M_n}{n!} = \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z)}{z^{n+1}} dz = \frac{1}{2\pi r^n \sqrt{e}} \int_{|\theta| \leq \pi} e^A d\theta, \quad (3.6)$$

where

$$A = \frac{1}{2} e^{2r \cos \theta} + r e^{i\theta} - n\theta i. \quad (3.7)$$

We divide the integral in (3.6) into two parts as

$$\int_{|\theta| \leq \pi} e^A d\theta = \int_{|\theta| \leq \theta_0} e^A d\theta + \int_{\theta_0 \leq |\theta| \leq \pi} e^A d\theta, \quad (3.8)$$

where

$$\theta_0 = \sqrt{\frac{2 \log n}{n}}.$$

Let $\Re(A)$ denote the real part of A , and $\Im(A)$ the imaginary part. It follows from (3.7) that

$$\begin{aligned} \Re(A) &= \frac{1}{2} e^{2r \cos \theta} \cos(2r \sin \theta) + r \cos \theta, \\ \Im(A) &= \frac{1}{2} e^{2r \cos \theta} \sin(2r \sin \theta) + r \sin \theta - n\theta. \end{aligned} \quad (3.9)$$

For the part $\int_{|\theta| \leq \theta_0} e^A d\theta$, we have

$$\begin{aligned}\Re(A) &= \frac{e^{2r} + 2r}{2} - \frac{n(2r+1)}{2} \theta^2 + O(nr^3 \theta_0^4), \\ \Im(A) &= O(nr^2 \theta_0^3).\end{aligned}\tag{3.10}$$

Substituting them into $e^A = e^{\Re(A) + i\Im(A)}$, we get

$$\int_{|\theta| \leq \theta_0} e^A d\theta = \exp\left(\frac{e^{2r} + 2r}{2}\right) \int_{|\theta| \leq \theta_0} e^{-m\theta^2} d\theta \left(1 + O(nr^2 \theta_0^3)\right),\tag{3.11}$$

where $m = (2r+1)n/2$. Note that

$$\int_x^\infty e^{-t^2} dt = o(e^{-x^2}), \quad \text{as } x \rightarrow \infty.$$

The integral in (3.11) can be estimated as follows

$$\int_{|\theta| \leq \theta_0} e^{-m\theta^2} d\theta = \frac{1}{\sqrt{m}} \left(\sqrt{\pi} - 2 \int_{\sqrt{m/n}}^\infty e^{-t^2} dt \right) = \sqrt{\frac{\pi}{m}} \left(1 + o(e^{-r})\right).\tag{3.12}$$

By (3.11) and (3.12), we find

$$\int_{|\theta| \leq \theta_0} e^A d\theta = \exp\left(\frac{e^{2r} + 2r}{2}\right) \sqrt{\frac{2\pi}{(2r+1)n}} \left(1 + O(nr^2 \theta_0^3)\right).\tag{3.13}$$

Now we estimate the integration $\int_{\theta_0 \leq |\theta| \leq \pi} e^A d\theta$. By (3.9), we have

$$\int_{\theta_0 \leq |\theta| \leq \pi} e^A d\theta \leq 2\pi \max_{\theta_0 \leq \theta \leq \pi} e^{\Re(A)} \leq 2\pi \exp\left(\frac{1}{2} e^{2r \cos \theta_0} + r\right).$$

Since

$$2r \cos \theta_0 = 2r - r\theta_0^2 + O(r\theta_0^4),$$

we get

$$\int_{\theta_0 \leq |\theta| \leq \pi} e^A d\theta = O\left(\exp\left(\frac{e^{2r}}{2} - \frac{n\theta_0^2}{2} + r\right)\right).$$

It is easy to check that

$$\lim_{n \rightarrow \infty} \frac{\exp\left(\frac{e^{2r}}{2} - \frac{n\theta_0^2}{2} + r\right)}{\exp\left(\frac{e^{2r} + 2r}{2}\right) \sqrt{\frac{2\pi}{(2r+1)n}} nr^2 \theta_0^3} = 0.\tag{3.14}$$

Namely, the remainder of $\left|\int_{\theta_0 \leq |\theta| \leq \pi} e^A d\theta\right|$ is smaller than the remainder of $\left|\int_{|\theta| \leq \theta_0} e^A d\theta\right|$.

By (3.13), we have

$$\int_{|\theta| \leq \pi} e^A d\theta = \exp\left(\frac{e^{2r} + 2r}{2}\right) \sqrt{\frac{2\pi}{(2r+1)n}} \left(1 + O(nr^2 \theta_0^3)\right).$$

Hence by (3.6) and Stirling's formula

$$n! = \frac{\sqrt{2\pi n} n^n}{e^n} \left(1 + O(n^{-1})\right),$$

we have

$$M_n = \frac{1}{\sqrt{2r+1}} \left(\frac{n}{r}\right)^n \exp\left(\frac{n}{2r} - n + r - 1\right) \left[1 + O\left(\frac{\log^{7/2} n}{\sqrt{n}}\right)\right]. \quad (3.15)$$

By Equation (2.4) and Lemma 2.1, we find

$$\left(\frac{n}{r}\right)^n = e^{2nr+r} \left(1 + O\left(\frac{\log^2 n}{n}\right)\right).$$

Together with (3.15), we arrive at (3.5). This completes the proof. \blacksquare

As will be seen in the next theorem, the remainder $O\left(\frac{\log^{7/2} n}{\sqrt{n}}\right)$ plays an essential role in estimating the variance $V(\xi_n)$.

Theorem 3.5. *We have*

$$E(\xi_n) = \frac{M_{n+1}}{2M_n} - 1 \sim \frac{n}{\log n}, \quad (3.16)$$

$$V(\xi_n) = \frac{M_{n+2}}{4M_n} - \frac{M_{n+1}^2}{4M_n^2} - \frac{1}{2} \sim \frac{n}{\log^2 n}. \quad (3.17)$$

Proof. It can be easily checked that the expectation and the variance of ξ_n can be expressed by

$$E(\xi_n) = \frac{M'_n(1)}{M_n},$$

$$V(\xi_n) = E(\xi_n) - E(\xi_n)^2 + \frac{M''_n(1)}{M_n}.$$

Thus we can deduce the exact formulas in (3.16) and (3.17). In view of Theorem 3.4, Lemma 2.1 and Lemma 2.3, we find

$$\frac{M_{n+1}}{2M_n} - 1 \sim \frac{n}{\log n}.$$

We now proceed to derive the approximation in (3.17). Suppose that

$$t_i(e^{2t_i} + 1) = n + i,$$

for $i = 0, 1, 2$. By Theorem 3.4, we have

$$\frac{M_{n+2}}{M_n} - \frac{M_{n+1}^2}{M_n^2} = \left(\sqrt{\frac{2t_0+1}{2t_2+1}}e^A - \frac{2t_0+1}{2t_1+1}e^B\right) \left(1 + O\left(\frac{\log^{7/2} n}{\sqrt{n}}\right)\right), \quad (3.18)$$

where

$$A = 4t_2 + \left(2nt_2 - 2nt_0 - 2 + \frac{1}{t_2}\right) - \left(\frac{n}{2t_0} - \frac{n}{2t_2}\right) + (2t_2 - 2t_0),$$

$$B = 4t_1 + \left(4nt_1 - 4nt_0 - 2 + \frac{1}{t_1}\right) - \left(\frac{n}{t_0} - \frac{n}{t_1}\right) + (4t_1 - 4t_0).$$

By Lemma 2.1, both $\sqrt{\frac{2t_0+1}{2t_2+1}}$ and $\frac{2t_0+1}{2t_1+1}$ can be estimated by $1 + O\left(\frac{1}{n \log n}\right)$. Because of the estimates in Lemma 2.3, (3.18) simplifies to

$$\frac{M_{n+2}}{M_n} - \frac{M_{n+1}^2}{M_n^2} = (e^A - e^B) \left(1 + O\left(\frac{\log^{7/2} n}{\sqrt{n}}\right)\right). \quad (3.19)$$

By Cauchy's mean value theorem, there exists a constant C such that $B < C < A$ and

$$e^A - e^B = (A - B)e^C. \quad (3.20)$$

On one hand, Lemma 2.3 yields

$$\begin{aligned} A - B &= \left(4t_2 - 4t_1 + \frac{1}{t_2} - \frac{1}{t_1}\right) - (2n + 2)(2t_1 - t_0 - t_2) + \frac{n}{2} \left(\frac{1}{t_0} + \frac{1}{t_2} - \frac{2}{t_1}\right) \\ &= \frac{1}{n} \left(1 + O\left(\frac{1}{n^2 \log n}\right)\right). \end{aligned} \quad (3.21)$$

On the other hand, by Lemma 2.1 we find that

$$e^C = \frac{4n^2}{\log^2 n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right). \quad (3.22)$$

Substituting (3.22) and (3.21) into (3.20), we deduce that

$$e^A - e^B = \frac{4n}{\log^2 n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right). \quad (3.23)$$

Substituting (3.23) into (3.19), we obtain the approximation of $V(\xi_n)$. This completes the proof. \blacksquare

By (3.17), we see that $V(\xi_n)$ tends to infinity as $n \rightarrow \infty$. Hence Theorem 3.1 follows from Theorem 3.3 and Proposition 3.2.

For B_n -partitions without zero-block, we have an analogous limiting distribution. Using the saddle point method as in the proof of Theorem 3.4, we obtain the following estimates of N_n .

Theorem 3.6. *We have*

$$N_n = \frac{1}{\sqrt{(2r_0 + 1)}} \exp\left(2nr_0 - n + \frac{n}{2r_0} - \frac{1}{2}\right) \left(1 + O\left(\frac{\log^{7/2} n}{\sqrt{n}}\right)\right) \quad (3.24)$$

$$\sim \frac{1}{\sqrt{\log n}} \exp\left(2nr_0 - n + \frac{n}{2r_0} - \frac{1}{2}\right). \quad (3.25)$$

where r_0 is the unique positive solution of the equation $r_0 e^{2r_0} = n$.

We remark that the approximation (3.25) can also be proved by Hayman's theorem [12].

Corollary 3.7. *We have*

$$\frac{N_n}{M_n} \sim \sqrt{\frac{\log n}{2n}}. \quad (3.26)$$

Proof. Let $r_0 e^{2r_0} = n$ and $r_1(e^{2r_1} + 1) = n$. By Theorem 3.4 and Lemma 2.1, we obtain that

$$M_n \sim \frac{2n}{\log^{3/2} n} \exp\left(2nr_1 - n + \frac{n}{2r_1} - 1\right).$$

Using (3.25), we get

$$\frac{N_n}{M_n} \sim \frac{\log n}{2n} \exp\left(2n(r_0 - r_1) - \frac{n(r_0 - r_1)}{2r_0 r_1} + \frac{1}{2}\right). \quad (3.27)$$

By Cauchy's mean value theorem, we have

$$n(r_0 - r_1) = \frac{r_0}{2} - \frac{1}{4} + O\left(\frac{1}{\log n}\right). \quad (3.28)$$

Thus (3.26) follows from (3.27) and (3.28). This completes the proof. \blacksquare

Recall that $N_{n,k}$ is the number of B_n -partitions without zero-block having k block pairs. It can be verified that for any $n \geq 1$, the polynomial

$$N_n(x) = \sum_k N_{n,k} x^k$$

has n distinct real roots. Let ξ'_n be the random variable of the number of block pairs in B_n -partitions without zero-block. Using the same argument as that for ξ_n , we find

$$\begin{aligned} E(\xi'_n) &= \frac{N_{n+1}}{2N_n} - 1 \sim \frac{n}{\log n}, \\ V(\xi'_n) &= \frac{N_{n+2}}{4N_n} - \frac{N_{n+1}^2}{4N_n^2} - \frac{1}{2} \sim \frac{n}{\log^2 n}. \end{aligned}$$

Hence $V(\xi'_n)$ tends to infinity as n does. By Proposition 3.2, we are led to the following assertion.

Theorem 3.8. *The limiting distribution of the random variable ξ'_n is normal.*

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